Fast Nielsen–Thurston Classification

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Algorithms in Complex Dynamics and Mapping Class Groups
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Nielsen–Thurston Classification Theorem

Any $[f] \in \mathrm{MCG}(S)$ has a representative which is either

1. periodic: $\phi^n = \mathrm{id}$ for some $n \neq 0$, or
2. reducible: $\phi(C) = C$ for some 1-submanifold C, or
3. pseudo-Anosov: these exists two transverse measured foliations $(\mathcal{F}^s, \mu^s)$,$(\mathcal{F}^u, \mu^u)$ and a number $\lambda > 1$

$$f(\mathcal{F}^s, \mu^s) = (\mathcal{F}^s, (1/\lambda)\mu^s)$$
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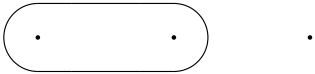

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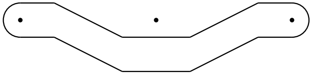

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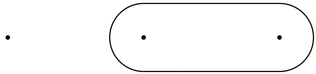

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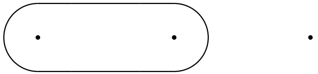

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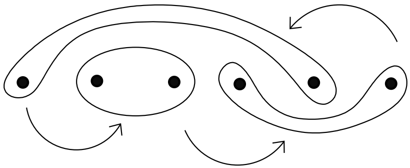

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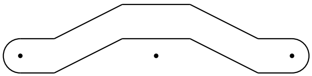

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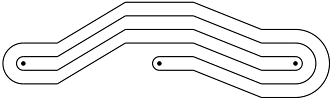

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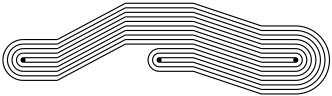

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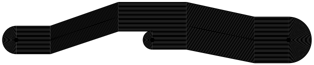

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Nielsen–Thurston Classification Problem

Given $[f] \in \mathrm{MCG}(S)$ *determine its Nielsen–Thurston type and,*

- *if periodic: find its order,*
- *if reducible: find its reducing curves,*
- *if pseudo-Anosov: find* $(\mathcal{F}^s, \mu^s)$*,* $(\mathcal{F}^u, \mu^u)$ *and* $\lambda > 1$*.*

History

- Higher genus surfaces
  - ⋆ Thurston (1970's)& Mosher (1982): exponential.
  - ⋆ Bestvina–Handel (1995): exponential, implemented by Toby Hall for D_n and Peter Brinkman for higher genus surfaces.
  - ⋆ Hamidi–Tehrani–Chen (1996): exponential.
  - ⋆ Koberda–Mangahas (2014): exponential.
  - ⋆ Bell–Webb (2016): NP and co–NP.
- Braids
  - ⋆ Los (1993): quadratic.
  - ⋆ Bernadete-Gutierrez–Nitecki (1995), Calvez (2013): quadratic time algorithm, Garside structure of B_n is used.

Main Theorem

Theorem in progress (Margalit–Taylor–Strenner–Y.)

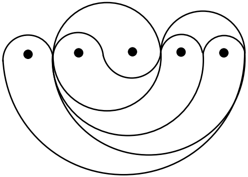
There exists a quadratic time algorithm to solve the Nielsen–Thurston classification problem.

Theorem (Bell–Webb)

Polynomial time algorithm to determine the Nielsen–Thurston classification type and find reducing curves, order and translation length in the curve complex.

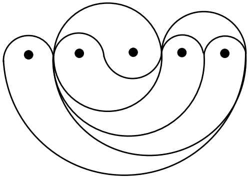
Basic Notions and Terminology

Throughout the talk we work on D_n.



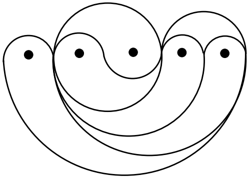
Basic Notions and Terminology

The results apply to any surface.

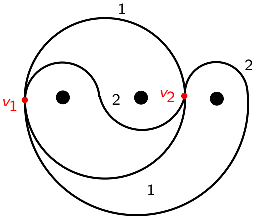


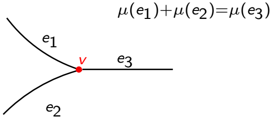
Basic Notions and Terminology

Throughout the talk standard train tracks will be used.



⋆ A measured train track is assigned with a transverse measure $\mu \in \mathcal{W}(\tau)$:





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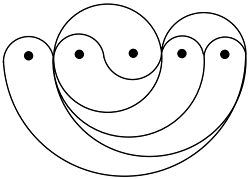
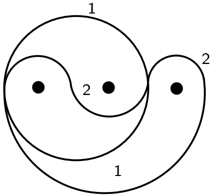


- ⋆ Curves/measured foliations are carried by τ if they arise from some transverse measure on τ:



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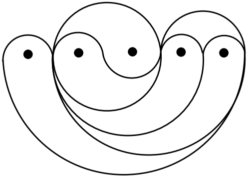

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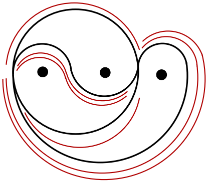

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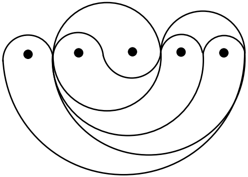

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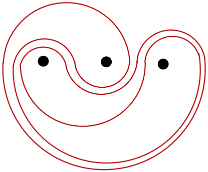

$C \prec \tau$

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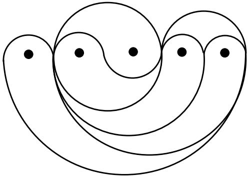

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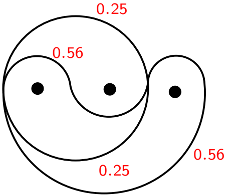

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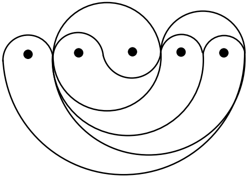

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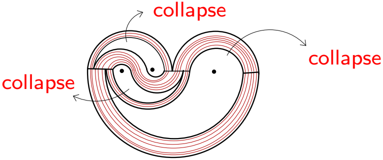

$$(\mathcal{F}, \mu) \prec \tau$$

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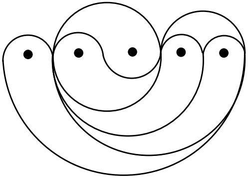

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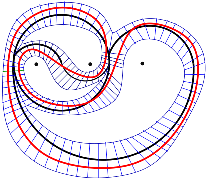

C and $\mathcal{F}$ can smoothly be embedded inside $\mathcal{N}(\tau)$:

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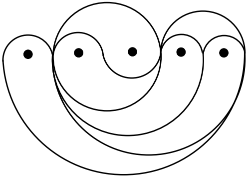

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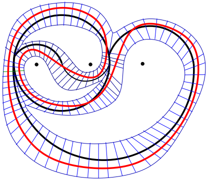

$\mathcal{MF}(\tau) \to \mathcal{W}(\tau)$ is a homeomorphism.

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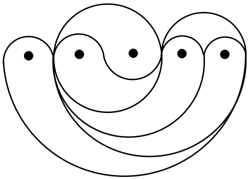

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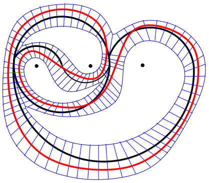

Train tracks define charts on $\mathcal{MF}$ and $\mathcal{PMF}$.

Basic Notions and Terminology

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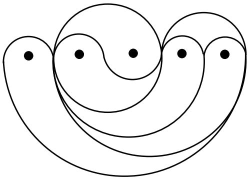

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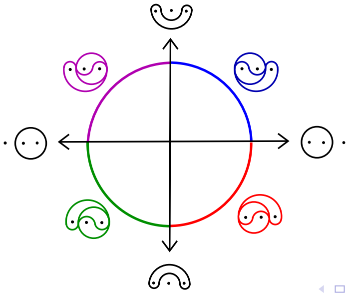

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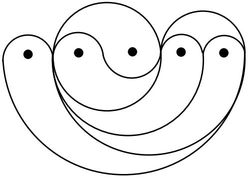
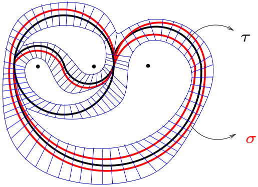


- ⋆ Similar definition when a train track is carried by another train track:



$\sigma \prec \tau$

Basic Notions and Terminology

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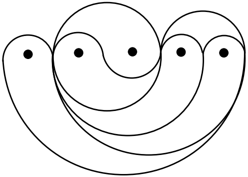
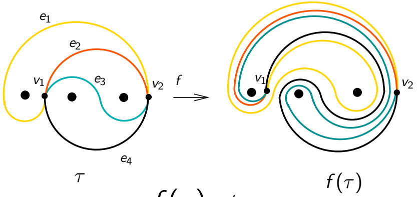


⋆ τ is invariant if $f(\tau) \prec \tau$.



$$f(\tau) \prec \tau$$

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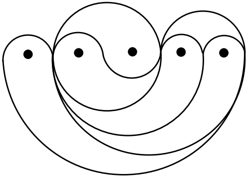
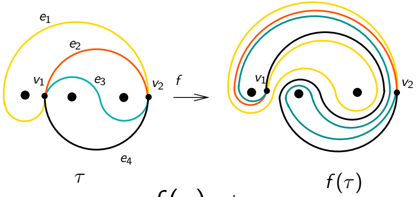


⋆ $\mathcal{W}(f(\tau)) \subseteq \mathcal{W}(\tau)$.



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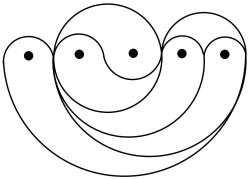
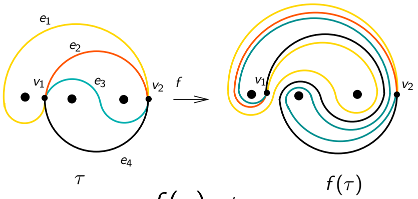


⋆ By Brouwer Fixed Point Theorem f has a fixed point in $\mathcal{W}(\tau)$.



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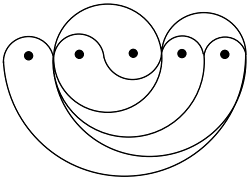

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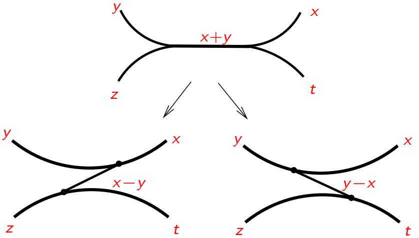

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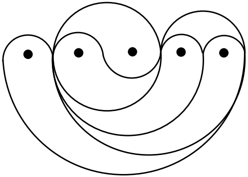

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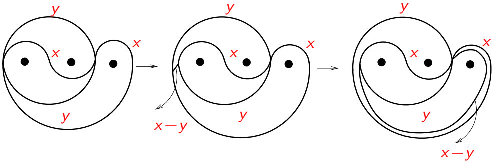
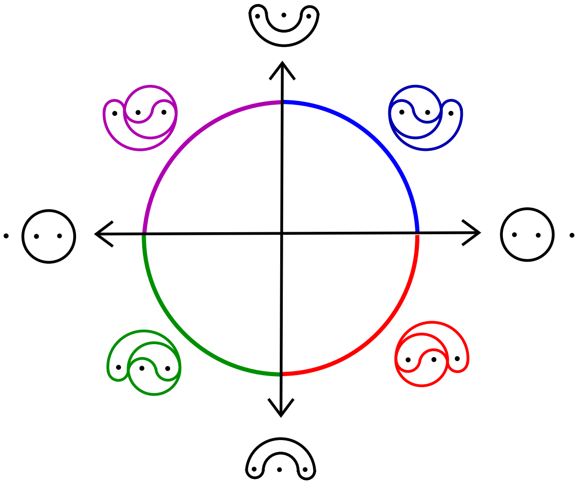
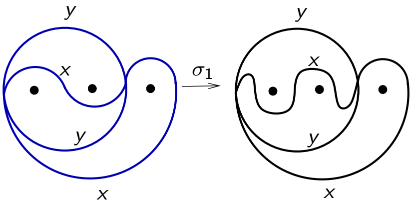


Charts on $\mathcal{PMF}$



MCG induce piecewise linear action on PMF.

Piecewise Linear Action on $\mathcal{PMF}$



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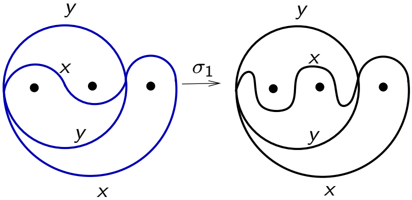
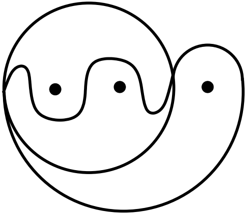


- **If** $x < y$



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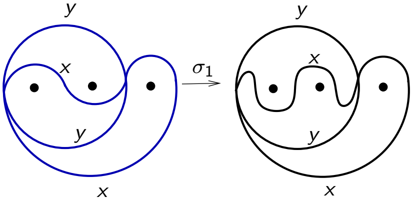

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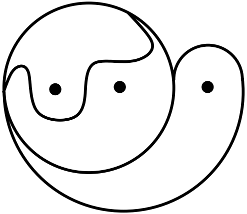

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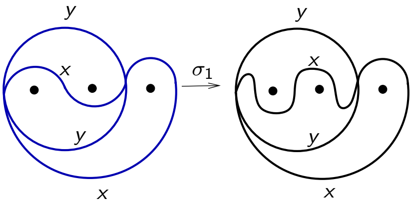

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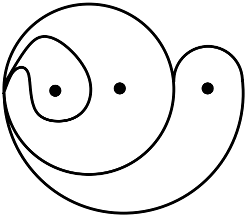

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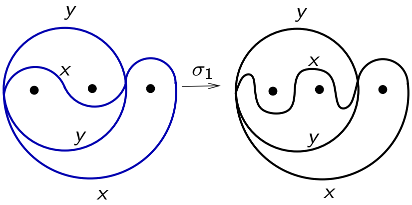

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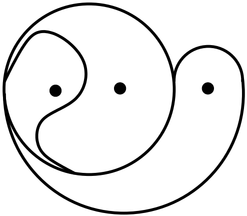

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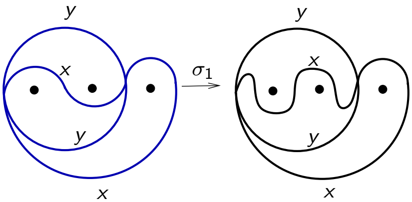
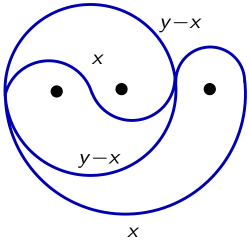


- **If** $x < y$ the blue chart is mapped back to itself.



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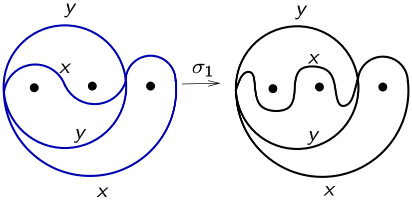

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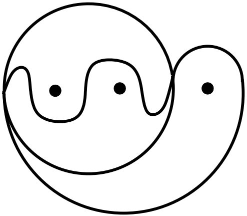

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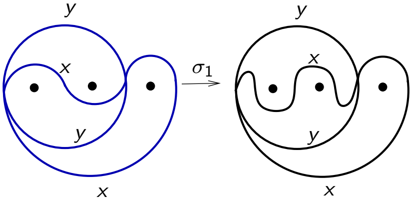

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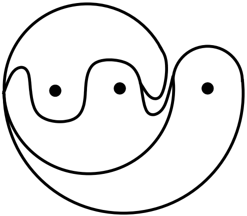

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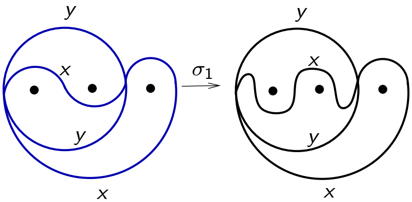

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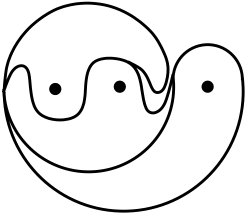

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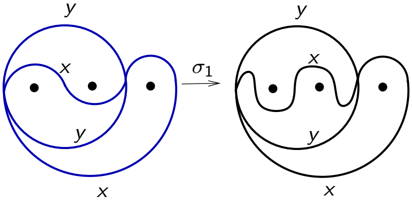

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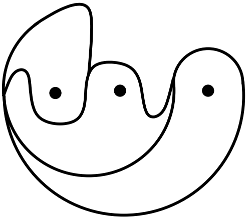

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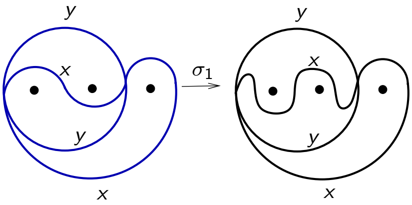

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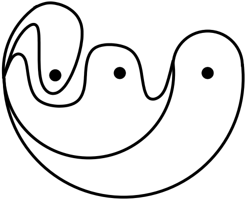

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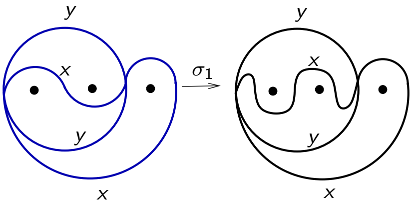

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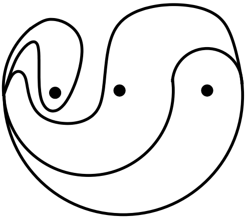

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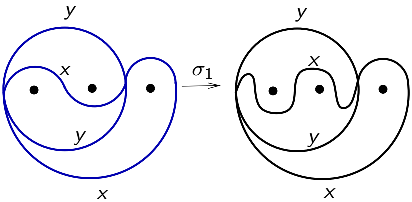

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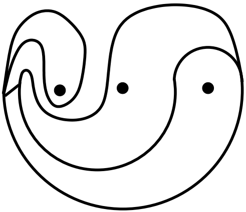

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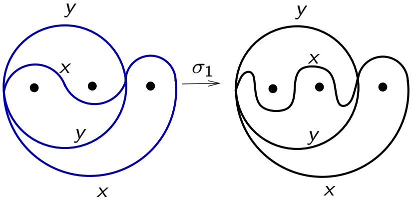

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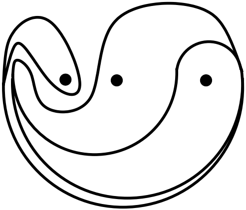

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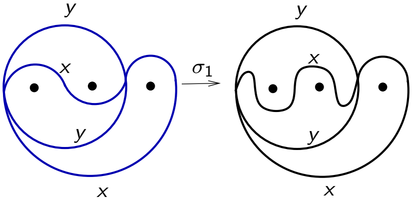

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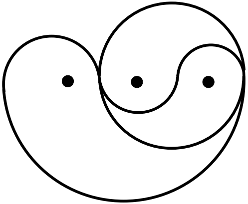

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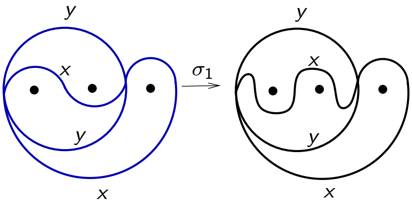

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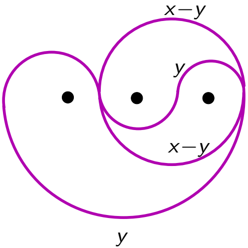
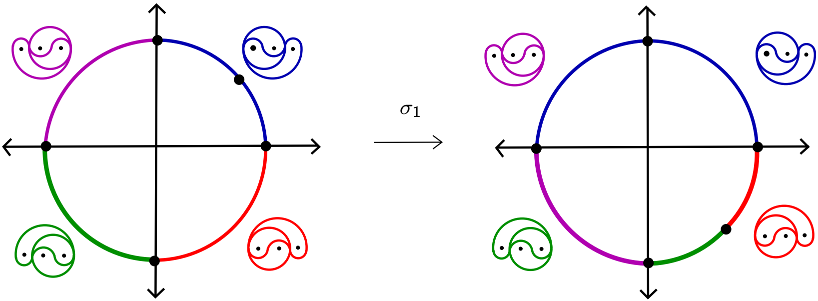


Piecewise Linear Action on $\mathcal{PMF}$

$\mathcal{MCG}$ induce piecewise linear action on $\mathcal{PMF}$.



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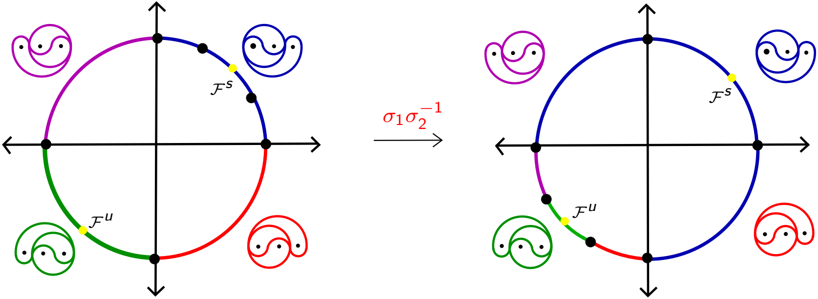
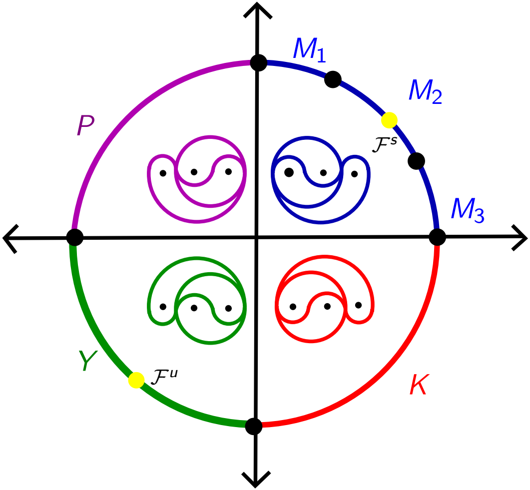


Action of a pseudo–Anosov mapping class on $\mathcal{PMF}$.

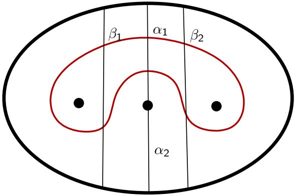
Piecewise Linear Action on $\mathcal{PMF}$

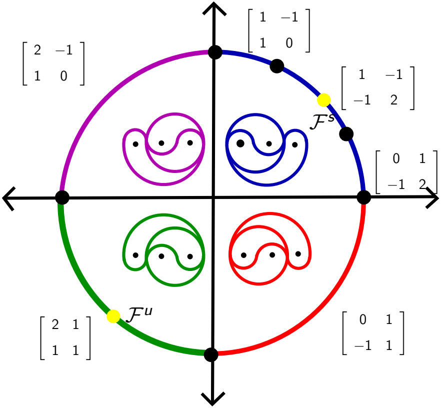


$$M_1 \to P \to Y \circlearrowleft, M_3 \to K \to Y \circlearrowleft, M_2 \to M$$

Dynnikov's coordinates

Other coordinates could also be used.





$$\rho(\mathcal{L}) = (-1; 0)$$

$$a_i = \frac{\alpha_{2i} - \alpha_{2i-1}}{2} \quad ; \quad b_i = \frac{\beta_i - \beta_{i+1}}{2}$$

$$a'_{i-1} = \left[a_{i-1}(1 + b_{i-1}) + a_i b_{i-1} \right],$$

$$b'_{i-1} = \left[\frac{a_i b_{i-1} b_i}{a_{i-1}(1 + b_{i-1})(1 + b_i) + a_i b_{i-1}} \right]$$

$$a'_i = \left[\frac{a_{i-1} a_i b_i}{a_{i-1}(1 + b_i) + a_i} \right],$$

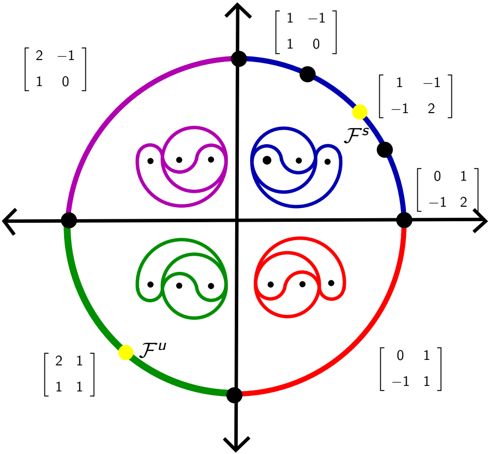
$$b'_i = \left[\frac{a_{i-1}(1 + b_{i-1})(1 + b_i) + a_i b_{i-1}}{a_i} \right];$$

$M_1 \to P \to Y$⮌, $M_3 \to K \to Y$⮌, $M_2 \to M$

pseudo–Anosov action on $\mathcal{PMF}$

Theorem (Thurston)

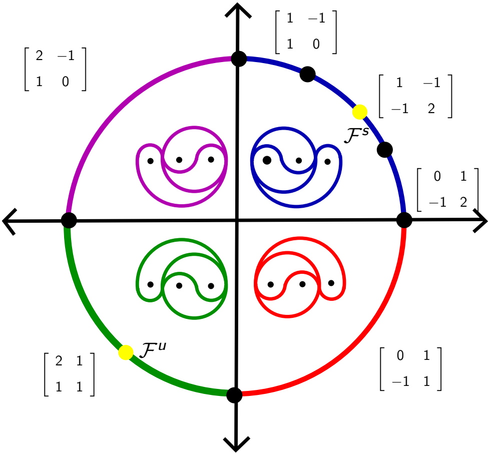
$\mathcal{PMF}$ has PIP structure (that is, locally described by integer matrices).



pseudo–Anosov action on $\mathcal{PMF}$

Theorem (Thurston)

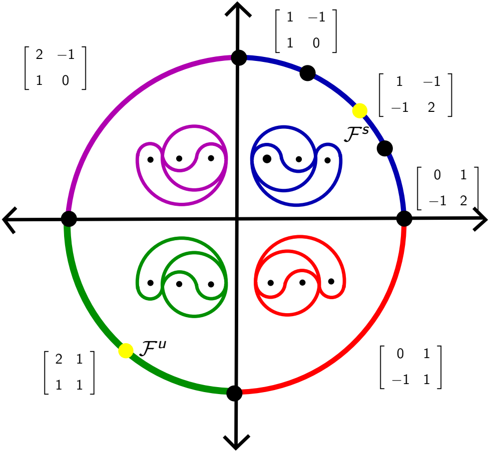
A pseudo-Anosov mapping class $[f]$ *has two fixed points, and both lie on* $\mathcal{PMF}$*:* $[\mathcal{F}^u, \mu^u]$ *and* $[\mathcal{F}^s, \mu^s]$.



pseudo–Anosov action on $\mathcal{PMF}$

Theorem (Thurston)

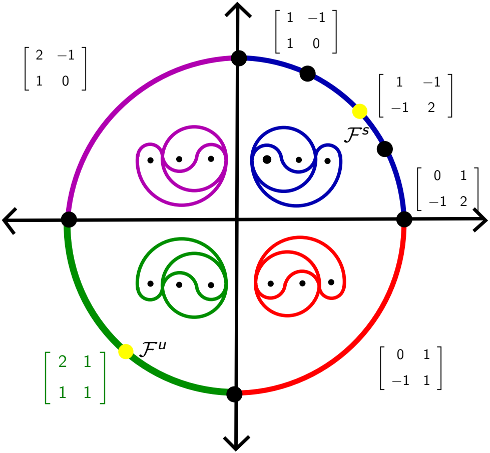
Every point (other than $[\mathcal{F}^s, \mu^s]$) converges to $[\mathcal{F}^u, \mu^u]$ on $\mathcal{PMF}$ under $[f]$.



pseudo–Anosov action on $\mathcal{PMF}$

Goal

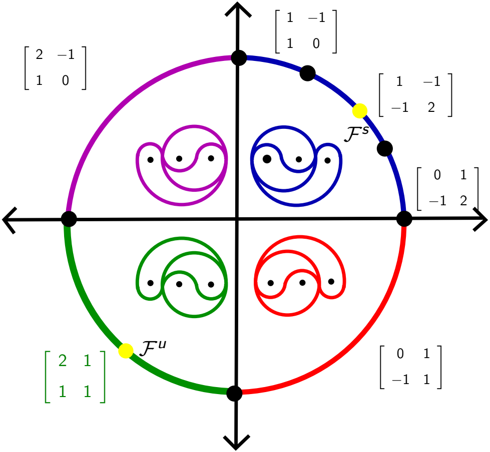
Compute the attracting linear region on $\mathcal{PMF}$, the piece which contains $[\mathcal{F}^u, \mu^u]$ and on which $[f]$ acts linearly.



pseudo–Anosov action on $\mathcal{PMF}$

Goal

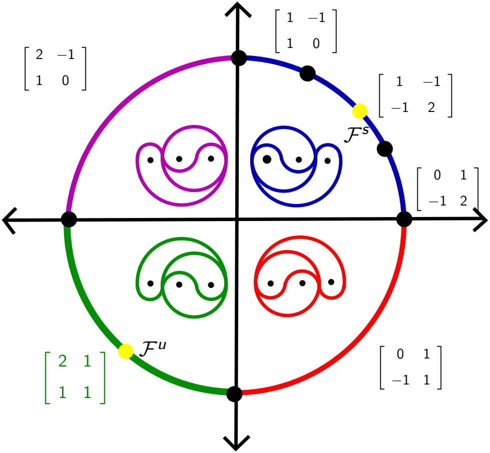
Compute the attracting matrix, the matrix acting on the attracting linear region.



pseudo–Anosov action on $\mathcal{PMF}$

Goal

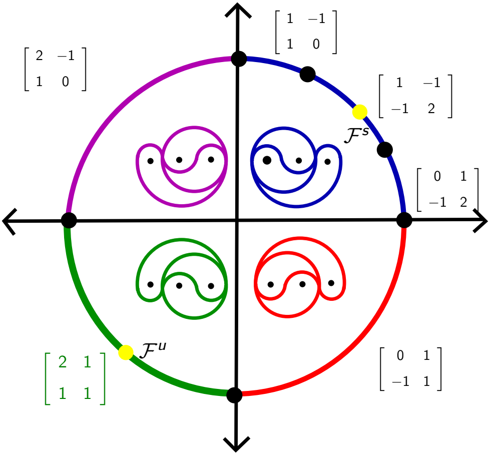
The way we actually find such attracting regions is to find an invariant train track.



pseudo–Anosov action on $\mathcal{PMF}$

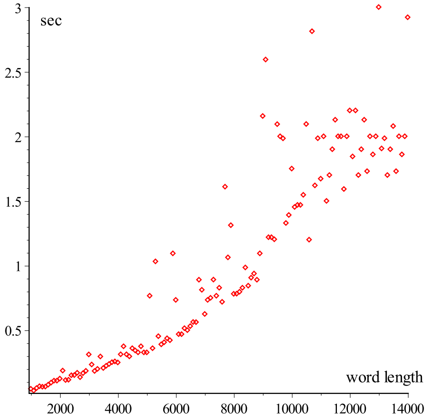
Approach

Use iteration.



How fast is it to reach attracting linear pieces?

Thurston : "Every other point. . . tends rather quickly toward the attracting point under iteration. . . "



Experiment results from Toby Hall's dynn.exe program

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Example

Take

$$\beta = \sigma_1^{-1}\sigma_2^{-3}\sigma_3^{-5}\sigma_1^{4}\sigma_2^{-2}\sigma_3^{-1}\sigma_1\sigma_2\sigma_3^{-2}(\sigma_2\sigma_3^{-2})^{19}\sigma_1^{-8}\sigma_3^{-1}\sigma_1^{-2}\sigma_2^{2}\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_3\sigma_1\sigma_2^{-1}\sigma_3^{-1}.$$

with $\lambda \approx 8.6 \times 10^{14}$.

- Train track program stops working.
- Attracting matrix is found in 0.001 seconds.

$$\begin{bmatrix} -68900596045753 & 200002959211464 & 146825523685804 & -943752747512 \\ -181490417757959 & 526825930446403 & 386751743244292 & -2485930314639 \\ -188609831321041 & 547491989409364 & 401923043417627 & -2583447121425 \\ 76020009608848 & -220669018174468 & -161996823859176 & 1041269554295 \end{bmatrix}$$

Experiment results from Toby Hall's dynn.exe program

How fast is it to reach invariant train tracks?

Theorem (Margalit–Taylor–Strenner–Y.)

Fix some family of standard train tracks $\{T_i\}$ and a curve c. Given a pA mapping class $[f]$ there is an invariant train track τ such that

- $\tau \prec T_i$ *for some* i, *and*
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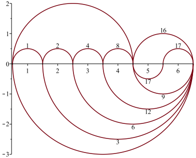

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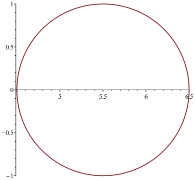

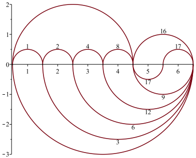

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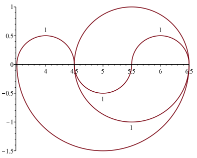

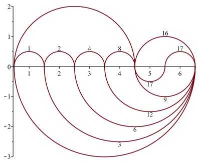

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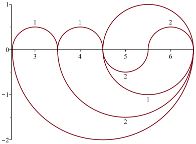

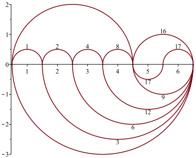

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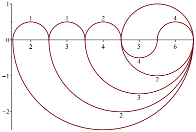

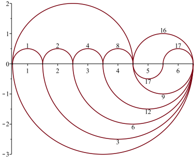

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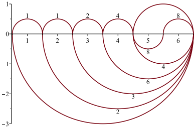
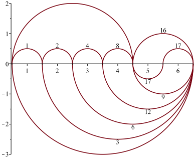


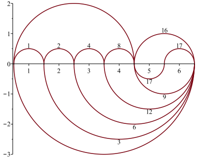


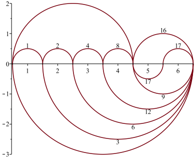
How fast is it to reach invariant train tracks?

Theorem (Margalit–Taylor–Strenner–Y.)

Fix some family of standard train tracks $\{T_i\}$ and a curve c. Given a pA mapping class $[f]$ there is an invariant train track τ such that

- $\tau \prec T_i$ *for some* i, *and*
- $f^Q(c) \prec \tau$ *for some* $1 \leq Q \leq (\mathit{constant})|\chi(S)|^2$ *(up to diagonal extension).*





How fast is it to reach invariant train tracks?

Theorem (Margalit–Taylor–Strenner–Y.)

Let $[f]$ be a pA mapping class. There is a constant $1 \leq Q \leq (constant)|\chi(S)|^2$ such that if T is any train track with $(\mathcal{F}^u, \mu^u) \prec T$ and $\text{slope}(c) \approx \text{slope}(T)$ there is an invariant train track τ such that

- $\tau \prec T$, *and*
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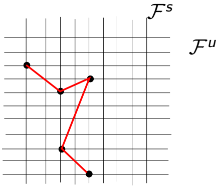
Proof. Key idea: "Slope of a curve" $\rightsquigarrow$ range of slopes of saddle connections when pulled tight in the flat structure.



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Let $[f]$ be a pA mapping class. There is a constant $1 \leq Q \leq (constant)|\chi(S)|^2$ such that if T is any train track with $(\mathcal{F}^u, \mu^u) \prec T$ and slope$(c) \approx$ slope(T) there is an invariant train track τ such that

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Proof. Key idea: "Slope of a curve" $\rightsquigarrow$ range of slopes of saddle connections when pulled tight in the flat structure.

F1. slope$(f^k(c)) = \lambda^{-2k}$ slope(c)

F2. If $\mathcal{F}^u \prec \tau$ and slope$(c) \ll$ slope(τ), then $c \prec \tau$ (up to diagonal extension).

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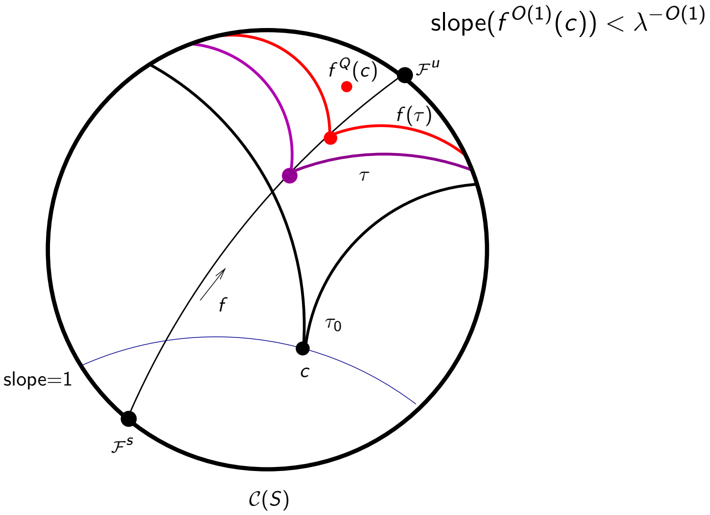

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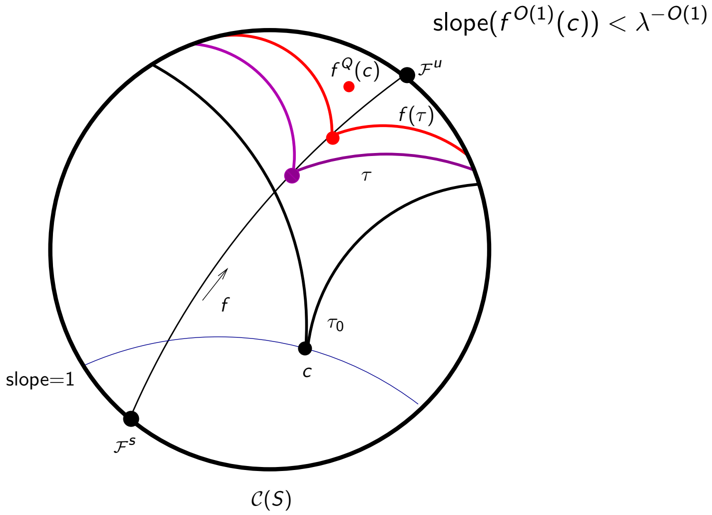
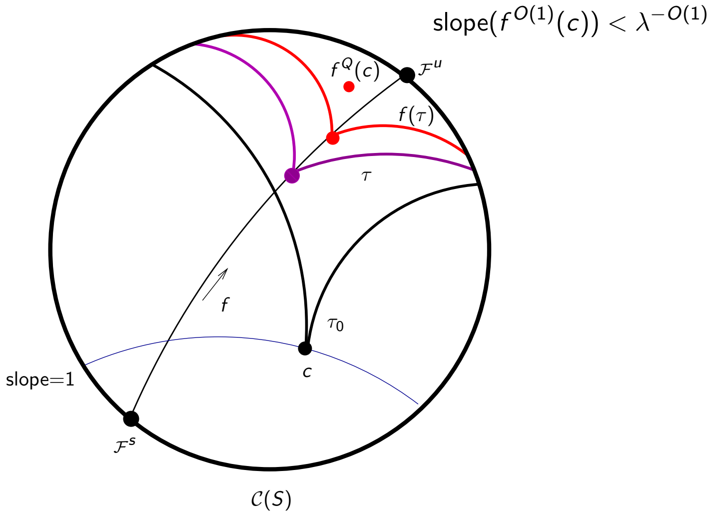


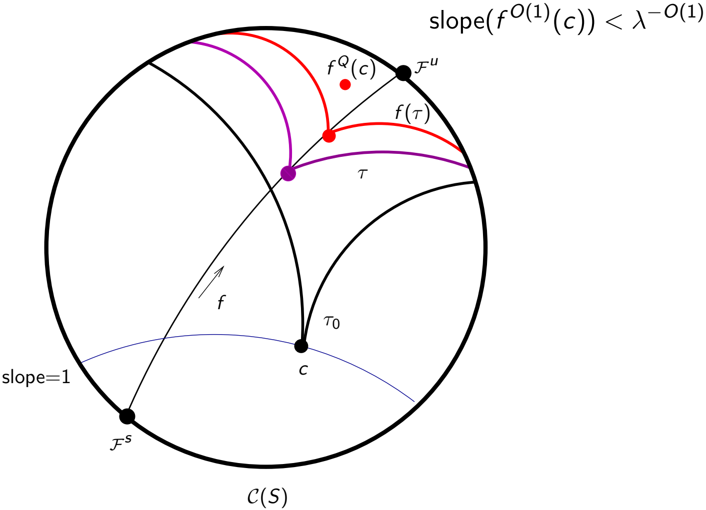
Agol cycle: If $(\mathcal{F}^u, \mu^u) \prec \tau_0$ and $\tau_0 \rightharpoonup \tau_1 \rightharpoonup \tau_2 \cdots$ is a maximal splitting sequence, then $\lambda\tau_{n+m} = f(\tau_n)$ for some n, m.

How fast is it to reach invariant train tracks?



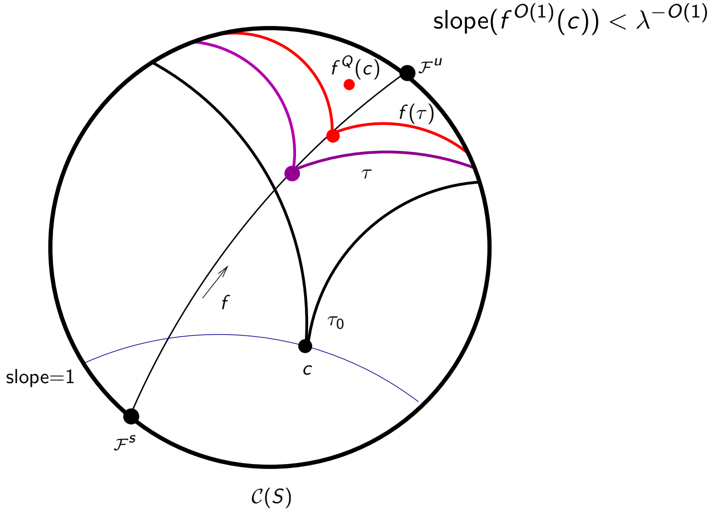
So τ_0 splits to Agol cyle.

How fast is it to reach invariant train tracks?



We use natural splitting sequences associated to f:

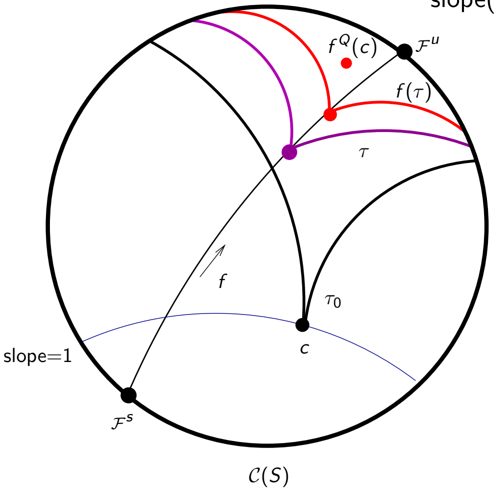
How fast is it to reach invariant train tracks?



Apply the generators one by one and split after each generator until the image train track is standard.

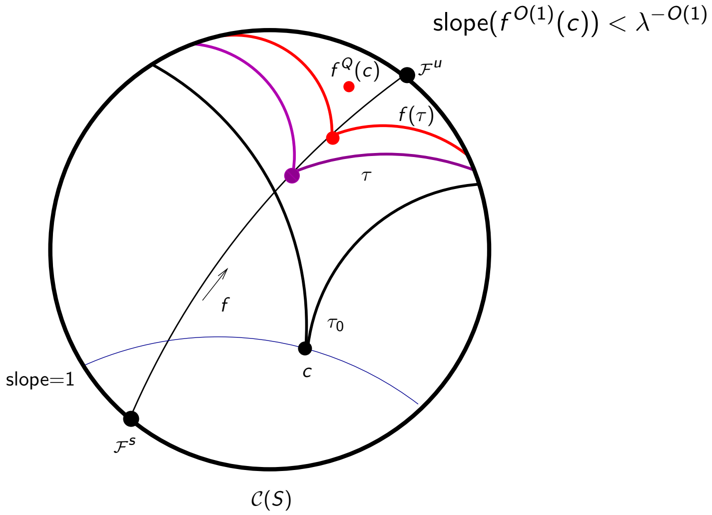
How fast is it to reach invariant train tracks?

$\text{slope}(f^{O(1)}(c)) < \lambda^{-O(1)}$



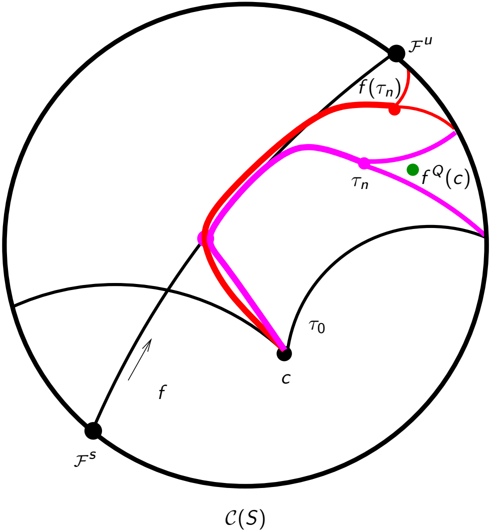
Split towards $f^Q(c)$.

How fast is it to reach invariant train tracks?



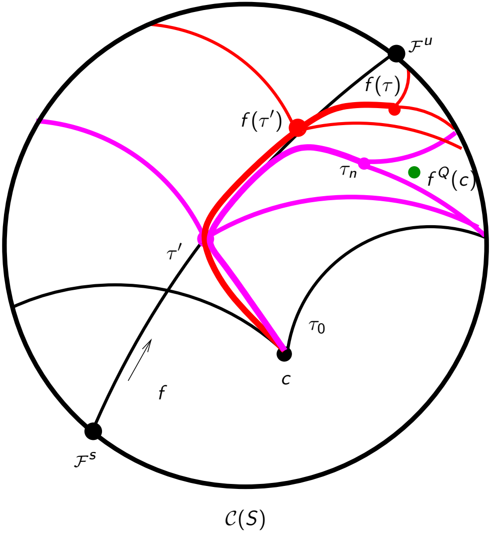
$\tau_0 \rightharpoonup \cdots \tau_n$ has length $O(n)$, $f(\tau_n)$ is carried by a standard train track and $f^Q(c) \prec \tau_n$.

How fast is it to reach invariant train tracks?



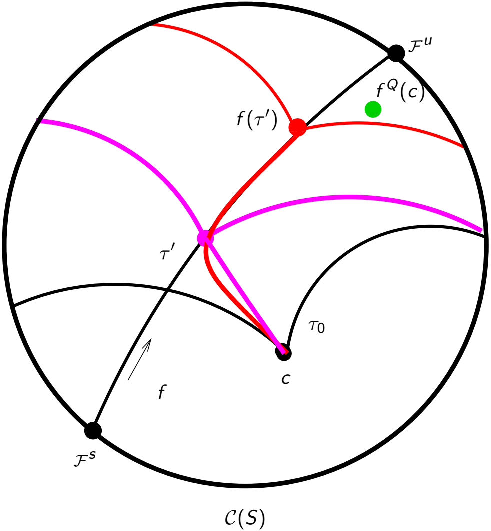
τ_n may not be invariant because of oversplitting.

How fast is it to reach invariant train tracks?



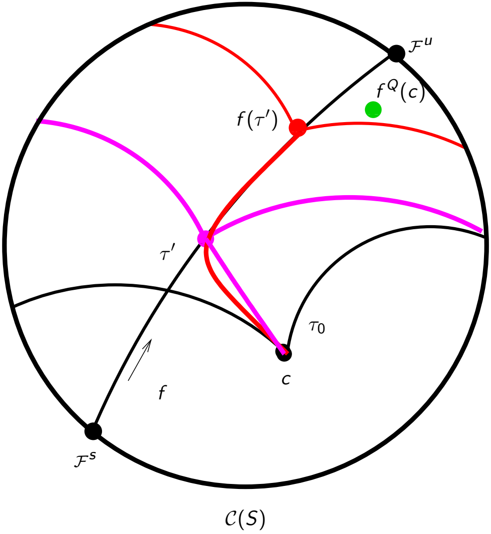
Fold τ_n until possible which gives τ'.

How fast is it to reach invariant train tracks?



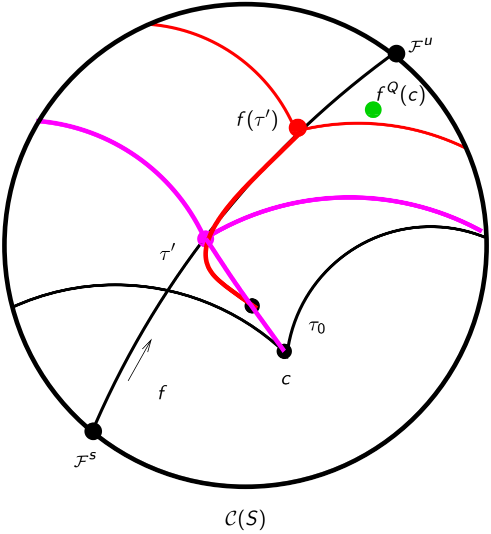
τ' is an invariant train track for f.

How fast is it to reach invariant train tracks?



To compute the graph map apply the same splitting sequence to τ_0 and check how $f(\tau')$ is carried.

How fast is it to reach invariant train tracks?



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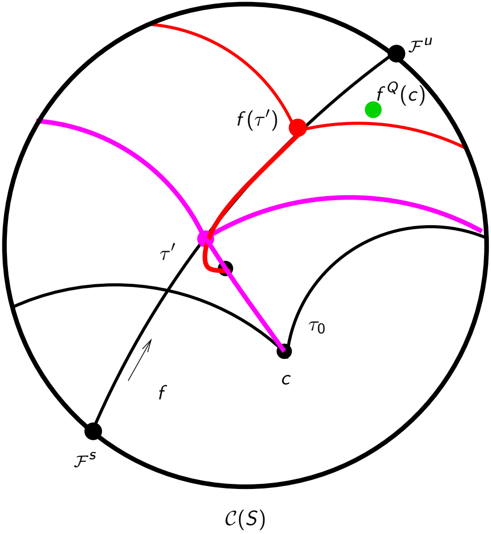

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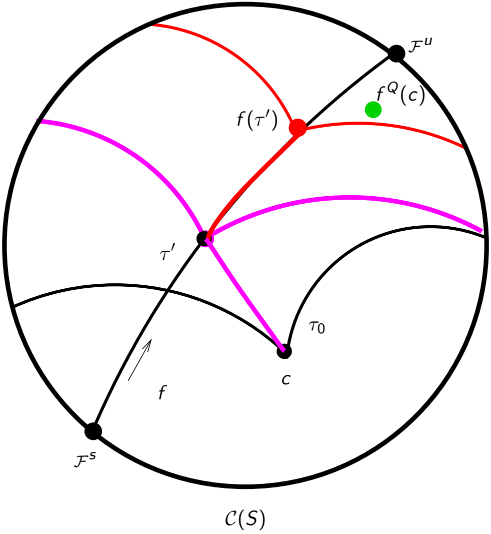
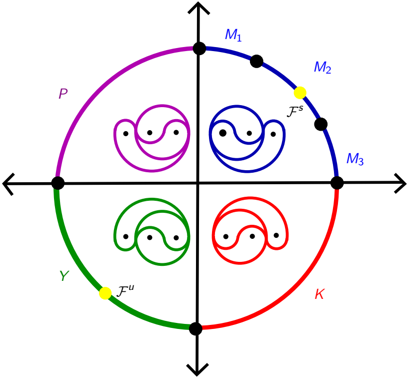


To compute the graph map apply the same splitting sequence to τ_0 and check how $f(\tau')$ is carried.

How fast is it to reach invariant train tracks?

Let $\beta = \sigma_1 \sigma_2^{-1}$.

- Take a curve.
- It takes at most 2 iterations to reach τ which carries $(\mathcal{F}^u, \mu^u)$.



$M_1 \to P \to Y\circlearrowleft, M_3 \to K \to Y\circlearrowleft, M_2 \to M$

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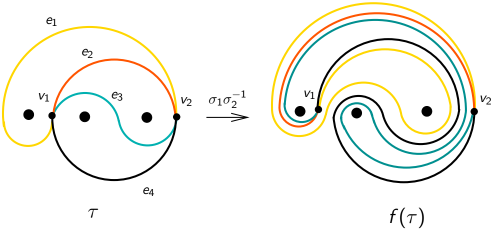

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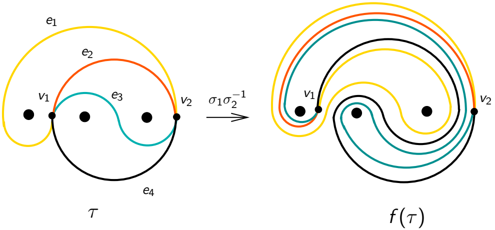

$$
\begin{aligned}
e_1 &\longrightarrow e_1 + e_2 + e_3 \\
e_2 &\longrightarrow e_1 \\
e_3 &\longrightarrow e_2 + e_3 + e_4 \\
e_4 &\longrightarrow e_1 + e_3 + e_4
\end{aligned}
$$

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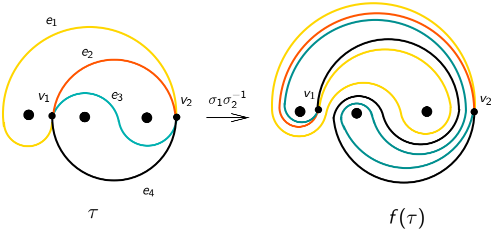

$$T = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \quad \lambda \approx 2.618$$

How fast is it to reach invariant train tracks?

Let $\beta = \sigma_1\sigma_1\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}$.

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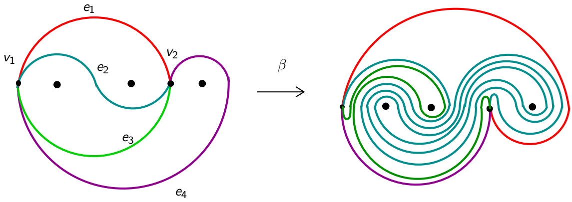

But τ is not invariant.

How fast is it to reach invariant train tracks?

Let $\beta = \sigma_1\sigma_1\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}$.

- Take a curve.
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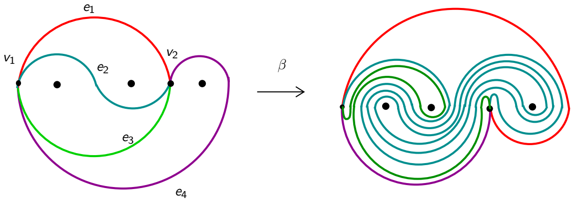
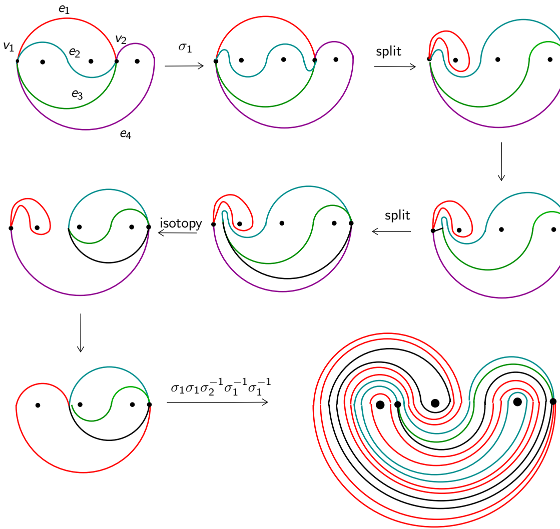


Solution: Split $\beta(\tau)$, apply the same splitting sequence to τ and compute how it is carried.

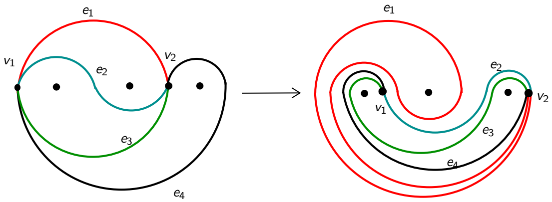
How fast is it to reach invariant train tracks?

A splitting sequence for $\beta(\tau)$:



How fast is it to reach invariant train tracks?

Same splitting sequence for τ:



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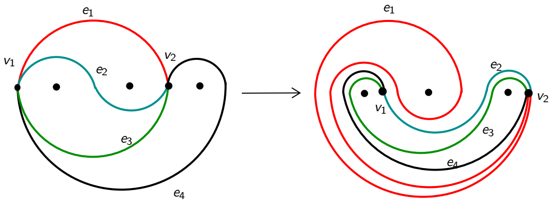
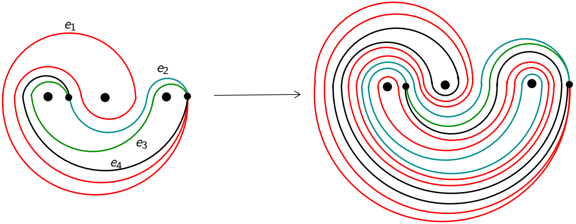


Compute the graph map and the transition matrix.



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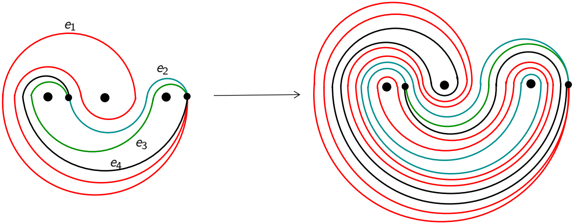

$$e_1 \longrightarrow 2e_1 + e_2 + e_3$$

$$e_2 \longrightarrow e_2 + e_3 + e_4$$

$$e_3 \longrightarrow e_2$$

$$e_4 \longrightarrow e_1 + e_2$$

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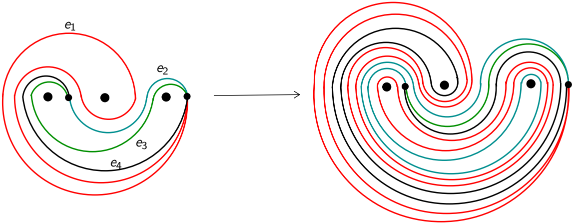
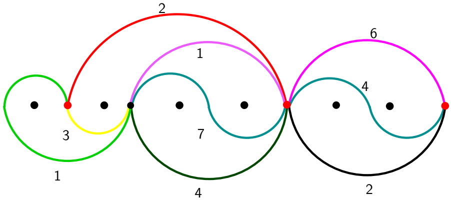


$$T = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \qquad \lambda \approx 2.618$$

How about reducibles?

Let $\beta = \sigma_1 \sigma_3 \sigma_1 \sigma_2^{-1} \sigma_4 \sigma_5^{-1} \sigma_3^{-1} \sigma_1^{-1} \in B_6$.



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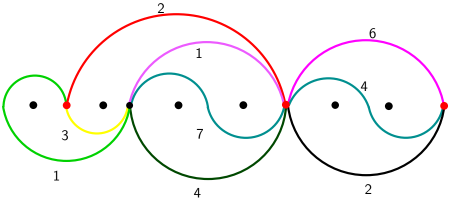
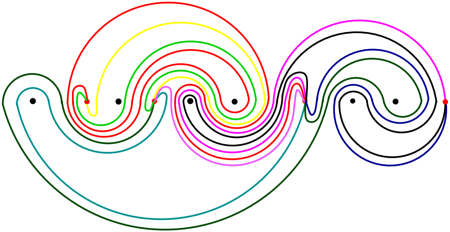


$f(\tau)$ is not invariant.



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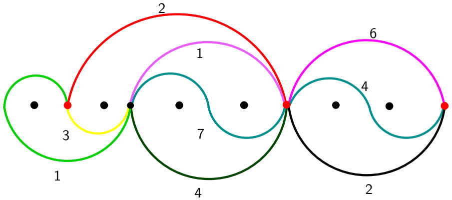
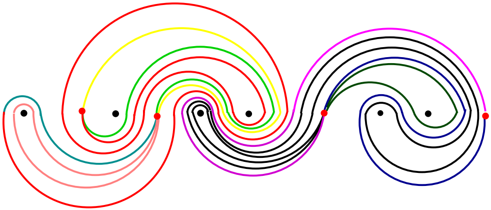


Reducing curves and partial pseudo-Anosovs appear after natural splitting sequence.



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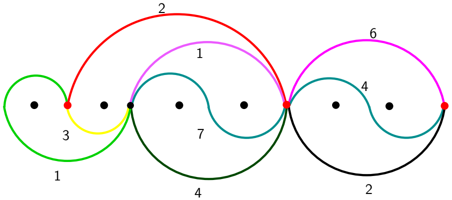

Work under progress.